



**Mathematics I – Semester 2 - 2025/2026**

Resit Assessment – 05th of June 2026

Duration:  $(120 + \varepsilon)$  minutes,  $|\varepsilon| \leq 30$

Version B

Name .....

ID Student #: .....

**Part I**

- Complete the following sentences in order to obtain true propositions. The items are independent from each other.
- There is no need to justify your answers.

(a) (12) Concerning the set:

$$S = \{x \in \mathbb{R} : |x - 4| < 2\} \cup \{7, 8, 9\}.$$

we may conclude that:

- $\text{cl}(S) = \dots\dots\dots$  (closure of  $S$ )
- The set of accumulation points of  $S$  is .....
- $\max S = \dots\dots\dots$  (maximum of  $S$ )
- The set  $S$  is a neighbourhood of  $x = \dots\dots$  (give an example of one element of  $S$ )

(b) (10) Let  $(g_n)_{n \in \mathbb{N}}$  be a **geometric progression** of positive terms such that  $g_2 = 6$  and  $g_3 = 2$ . Then:

- The ratio of  $(g_n)_{n \in \mathbb{N}}$  is  $r = \dots\dots\dots$  and  $g_1 = \dots\dots\dots$
- $g_n = \dots\dots\dots$  (general term of  $(g_n)_{n \in \mathbb{N}}$ )
- $\sum_{n=2}^{+\infty} g_n = \dots\dots\dots$

(c) (9) Consider the real maps  $f$  and  $g$  such that

$$f(x) = \frac{x-2}{x+1}, \quad x \neq -1 \quad \text{and} \quad g(x) = \sqrt{x}, \quad x \geq 0.$$

Then:

- The domain of  $g \circ f$  is  $D_{g \circ f} = \dots\dots\dots$
- $(g \circ f)'(3) = \dots\dots\dots$
- The equation of the horizontal asymptote of the graph of  $f$  as  $x \rightarrow +\infty$  is  
 $\dots\dots\dots$

(d) (6) Let  $a \in \mathbb{R}$ . Consider a twice differentiable map  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that its second-order Taylor polynomial centered at  $x = a$  is:

$$P_2(x) = 5 - 3(x - a) + 4(x - a)^2.$$

Then:

- $f''(a) = \dots\dots\dots$
- At  $x = a$ , the graph of  $f$  has its concavity turned  $\dots\dots\dots$  (upwards/downwards).

(e) (9) Consider a differentiable map  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that the tangent line to its graph at the point  $x = 3$  is defined by the equation  $y = -2x + 10$ . Then:

- $f(3) = \dots\dots\dots$  and  $f'(3) = \dots\dots\dots$
- Using the linear approximation,  $f(3.05) \approx \dots\dots\dots$

(f) (9) Consider the map  $F : \mathbb{R} \rightarrow \mathbb{R}$  defined by:

$$F(x) = \int_0^{x^2} e^{-t} dt$$

Hence:

- $\lim_{x \rightarrow +\infty} F(x) = \dots\dots\dots$
- $F'(x) = \dots\dots\dots$
- $F$  is monotonically  $\dots\dots\dots$  in the interval  $]0, +\infty[$ .

(g) (12) In the vector space  $\mathbb{R}^3$ , consider the vectors  $\vec{u} = (2, -1, 3)$ ,  $\vec{v} = (1, 2, 0)$ , and  $\vec{w} = (k, 1, -1)$ , where  $k \in \mathbb{R}$ . We may conclude that:

- $\vec{u} \cdot \vec{v} = \dots\dots\dots$  and  $\|\vec{u}\| = \dots\dots\dots$
- The value of  $k$  for which  $\vec{u}$  and  $\vec{w}$  are **orthogonal** is  $k = \dots\dots\dots$
- If  $k = 1$ , then  $2\vec{u} - 3(\vec{v} + \vec{w}) = (\dots, \dots, \dots)$ .

(h) (14) Consider the matrix  $\mathbf{M} = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \in M_{2 \times 2}(\mathbb{R})$  and  $n \in \mathbb{N}$ . Therefore:

- The zeros of  $\det(\mathbf{M} - \lambda \mathbf{I})$ , where  $\mathbf{I}$  is the identity matrix in  $M_{2 \times 2}(\mathbb{R})$ , are  

$$\lambda_1 = \dots\dots\dots \quad \text{and} \quad \lambda_2 = \dots\dots\dots$$
- $\det(\mathbf{M}^n) = \dots\dots\dots$  (express in terms of  $n$ ).
- $(\mathbf{M}^T)^{-1} = \begin{bmatrix} \dots & \dots \\ \dots & \dots \end{bmatrix}$ .

(i) (9) Consider the linear system  $\mathbf{A}X = \mathbf{B}$  with 3 variables  $x_1, x_2, x_3$ . Let  $\mathbf{A}$  be a  $4 \times 3$  matrix and  $[\mathbf{A}|\mathbf{B}]$  its augmented matrix.

- If  $r(\mathbf{A}) = 2$  and  $r(\mathbf{A}|\mathbf{B}) = 2$ , the system is classified as ..... and its degree of freedom is .....
- For the system to be **consistent and defined** (unique solution), the rank  $r(\mathbf{A})$  would have to be .....

## Part II

- Give your answers in exact form.
- In order to receive credit, you must show all of your work. If you do not indicate the way in which you solve a problem, you may get little or no credit for it, even if your answer is correct.

1. Using the **Principle of Mathematical Induction**, prove the following equality for all  $n \in \mathbb{N}$ :

$$\sum_{i=1}^n (2i - 1) = n^2$$

2. Consider the **continuous** function  $f : \mathbb{R} \rightarrow \mathbb{R}$  defined by:

$$f(x) = \begin{cases} \frac{\sqrt{1+kx}-1}{x} & \text{if } x > 0 \\ 1 & \text{if } x = 0 \\ e^{ax} + bx & \text{if } x < 0 \end{cases}$$

where  $a, b, k \in \mathbb{R}$ .

- Show that  $k = 2$ .
- Determine the relationship between  $a$  and  $b$  such that  $f$  is **differentiable** at  $x = 0$ .
- Assume  $a = 2$  and  $b = 1$ . Find the value  $c \in (-1, 0)$  such that:

$$f'(c) = \frac{f(0) - f(-1)}{1}$$

3. Consider the function  $f(x) = \frac{x}{\sqrt{1+x^2}}$ ,  $x \in \mathbb{R}$ .

(a) Using the **substitution**  $u = 1 + x^2$ , compute  $\int f(x) dx$ .

(b) Let  $R$  be the region bounded by the graph of  $f$ , the  $x$ -axis, and the vertical lines  $x = 0$  and  $x = \sqrt{3}$ . Calculate the exact value of the area of  $R$ .

4. Consider the matrix  $\mathbf{M} \in M_{3 \times 3}(\mathbb{R})$  and the matrix  $\mathbf{P} \in M_{3 \times 3}(\mathbb{R})$  defined by:

$$\mathbf{M} = \begin{bmatrix} \lambda & 1 & 1 \\ 1 & \lambda & 1 \\ 1 & 1 & \lambda \end{bmatrix} \quad \text{and} \quad \mathbf{P} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

where  $\lambda \in \mathbb{R}$ .

(a) Determine the values of  $\lambda$  for which the columns of  $\mathbf{M}$  are **linearly dependent**.

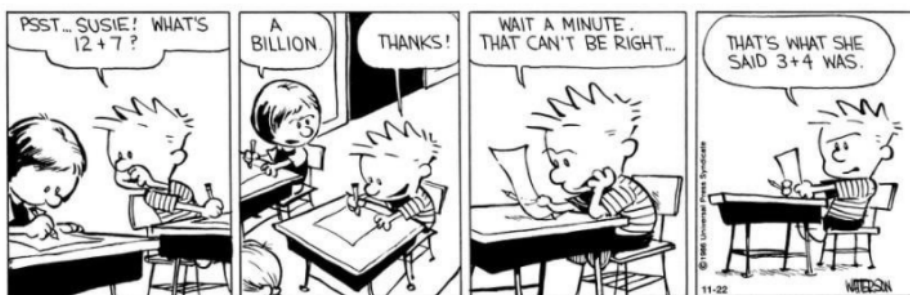
(b) Show that  $\mathbf{P}$  is invertible and calculate  $\det(2\mathbf{P}^{-1} \cdot \mathbf{M}^T)$  in terms of  $\lambda$ .

(c) Find the values of  $\lambda$  for which the system  $(\mathbf{M} - \mathbf{M}^T)\mathbf{v} = \mathbf{0}$ ,  $\mathbf{v} \in \mathbb{R}^3$ , has infinitely many solutions.

5. Consider the following system of linear equations with parameters  $a, b \in \mathbb{R}$  and variables  $x, y, z$ .

$$\begin{cases} (a-1)x + y + z = 1 \\ x + (a-1)y + z = b \\ x + y + (a-1)z = b^2 \end{cases}$$

Classify the system (consistent, inconsistent, unique solution, or infinitely many solutions) according to the values of the parameters  $a$  and  $b$ .



Scores:

I	II.1	II.2(a)	II.2(b)	II.2(c)	II.3(a)	II.3(b)	II.4(a)	II.4(b)	II.4(c)	II.5
90	10	10	10	10	10	10	10	10	10	20